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ALMOST EINSTEIN MANIFOLDS OF NEGATIVE RICCI CURVATURE

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1. Introduction

For a Riemannian manifold (M^n, g), denote by K its sectional curvature, a function on the Grassmannian bundle G_2M of 2-planes in the tangent bundle TM, by ρ its Ricci curvature, regarded as a function on the sphere bundle SM of unit tangent vectors, and by R its scalar curvature, which is a function on M. We normalize our curvature functions so that the sphere S^n of radius 1 has $K = 1$, $\rho = n - 1$ and $R = n(n-1)$. If M is compact, we denote by d its diameter, and by V its volume, and we define $r = ⨍ R = \frac{1}{V}\int R$ to be the average scalar curvature. (M^n, g) is called *Einstein* if the Ricci curvature ρ is a constant function $= \frac{r}{n}$.

This paper is concerned with compact almost Einstein manifolds of negative average scalar curvature, where ρ is almost a constant and $r < 0$.

For $n \geq 3$ and $\Lambda > 0$, we define $\mathscr{M}^-(n, \Lambda)$ to be the set of all smooth compact Riemannian manifolds (M^n, g) of dimension n, satisfying the following curvature bounds:

(i) $r < 0$,
(ii) $d^2 \max |K| \leq \Lambda^2$.

It is well known [4] that if $n \geq 3$, then any smooth compact manifold M^n admits a metric g with $r < 0$.

The main result of this paper is the following pinching theorem for the Ricci curvature.

Theorem. *For any $n \geq 3$ and $\Lambda > 0$, there exists an $\varepsilon(n, \Lambda) > 0$, depending only on n and Λ, such that if $(M^n, g) \in \mathscr{M}^-(n, \Lambda)$ and if its Ricci curvature satisfies*

$$\max_{SM} |n\rho/r - 1| < \varepsilon(n, \Lambda),$$

then M admits an Einstein metric $\overline{g}$ with $\rho(\overline{g}) \equiv -1$.

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In three dimensions, the Einstein metric $\overline{g}$ is of course a hyperbolic metric of constant negative sectional curvature. The basic technique used to prove the above theorem is to deform the metric in the direction of its Ricci curvature as was first successfully done by R. S. Hamilton in his fundamental work [9]. This flow of metrics, which by [9] exists, at least for a short positive time, determines a nonlinear parabolic partial differential equation for the curvature of the metric. In contrast to most previous work that followed Hamilton's flow (e.g. [10], [11]), which were mainly concerned with manifolds that are so positively curved that the flow leads to a spherical metric (or in dimension 4 to a locally symmetric metric of nonnegative curvature), we have here the more generic situation of a metric with negative Ricci curvature and we make no particular assumptions on the sectional curvature although we are dealing with all dimensions. On the other hand, our deformation is a small one and the final Einstein metric which we obtain is in fact near the initial metric which we started out with. Our result should therefore be regarded as a statement on the structural stability of the Einstein equation with a negative constant.

The main new ingredient in our proof is that, instead of the ordinary C^0-version of the Bochner-Weitzenböck formula for the curvature evolution equation and the ensuring argument involving the maximum principle which usually requires that the curvature is positive in some rather strong sense, we make use here of a much weaker L^2-estimate for the covariant Laplacian acting on the curvature tensor and its irreducible components, in particular, its trace free Ricci part, which are regarded as 2-forms with values in the endomorphisms of the tangent bundle. This general L^2-formula is valid for any compact Riemannian manifold and is based on the second Bianchi identity for the curvature. These L^2-versions of the Weitzenböck formula are used in the study of cohomology groups associated to cocompact discrete subgroups of real semisimple Lie groups (see, e.g., [14]). In fact we are led to such an L^2-formula by our interpretation of Hamilton's Ricci flow as a Yang-Mills type flow for Cartan connections of hyperbolic type as explained in a joint paper with E. A. Ruh [15].

In the present paper we will work in the better known framework of conventional Riemannian geometry and derive our basic L^2-formula in §2. This, together with our assumption that $r < 0$, proves that we have, in the L^2-sense, an exponential decay for the deviation from an Einstein metric along the flow. This estimate is then boot-strapped to a C^0-estimate in §3 using the powerful iteration technique due to J. Moser. Moser's method is perfectly suited for global estimates on a Riemannian manifold since it involves only the Sobolev imbedding constant and hence only the isoperimetric constant and this can be estimated in terms of a lower bound for the Ricci curvature and upper bound on the diameter. In particular we do not need any assumptions on the injectivity radius.

After having a C^0-estimate established, higher order estimates and the exponential convergence of the flow to a nearby Einstein metric then follow by standard interior regularity results for parabolic partial differential equations, since the nonlinearity in the evolution equations for the curvature involves only zeroth order terms.

It should be remarked that our method of proof fails if the manifold is almost Einstein with respect to a nonnegative Einstein constant. In particular, it is still an open question whether almost flat compact nilmanifolds admit an Einstein metric. It is known (see for example [4]) that they do not admit any Einstein metric with a nonnegative Einstein constant. This shows that the analogous statement to our result for the case where the Einstein constant is zero is, in general, false. For almost Einstein manifolds with a positive Einstein constant all the known results are geared towards proving a sphere theorem and hence usually assume some stronger assumption on the whole curvature tensor. Finally, since every compact 3-manifold is now known to carry a metric of negative Ricci curvature ([7], [5]), one cannot allow the ε in our theorem to be too large.

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2. L^2-estimates

We follow R. S. Hamilton's fundamental paper [9] and consider the evolution equation

$$\dot{g} = \frac{d}{dt} g = -2\,\mathrm{Rc} + \frac{2}{n} r(0) g, \tag{2.01}$$

where the dot on g signifies the time derivative $\frac{d}{dt}$, $\mathrm{Rc} = R_{ij}$ is the Ricci tensor of g and $r(0)$ is the average scalar curvature of the given initial metric $g(0)$ at time $t = 0$. This differs in normalization from the equation used by Hamilton [9] since $r(0)$ is constant in time.

It is proved in [9] that this flow of metrics can be integrated on a compact manifold for some maximal time interval $[0, T)$ and that if $T < \infty$, then $\lim_{t \to T} \max |K(t)| = \infty$.

We will freely adopt here the notation of [9] except for some minor modifications as in our paper [15]. For example, the Laplacians used here will always be nonnegative operators, which is opposite the sign convention of [9].

We denote by h the symmetric $(0, 2)$-tensor

(2.02) $$h = \mathrm{Rc} - \frac{r(0)}{n} g,$$

so that the basic evolution equation (2.01) becomes $\dot{g} = -2h$.

If μ denotes the volume form of g, then we have

(2.03) $$\dot{\mu} = -\operatorname{tr} h \mu = -\tilde{R}\mu,$$

where the trace of h is $\tilde{R} = R - r(0)$, the *hyperbolic scalar curvature.* The rate of change of the total volume V is then given by

(2.04) $$\frac{d}{dt} \log V(t) = -\frac{1}{V} \int \tilde{R}\mu.$$

By a slight modification of Corollary 7.5 of [9], we obtain the following evolution equation for the hyperbolic scalar curvature:

(2.05) $$\frac{\partial}{\partial t}\tilde{R} + \Delta\tilde{R} = 2|h|^2 + \frac{2}{n} r(0)\tilde{R}.$$

Henceforth we will use the symbols $|\ |$ and $\langle\ ,\ \rangle$ for the natural norm and scalar product induced by the metric g (at a given time t) on all tensors. We will also raise and lower indices using the metric and, following Einstein, sum over repeated indices; e.g., $|h|^2 = \operatorname{tr} h^2 = g^{ij} g^{kl} h_{ik} h_{jl}$.

(2.05) already indicates that the scaling term $-\frac{r(0)}{n} g$ used in our metric flow gives rise to the coefficient $\frac{2}{n} r(0)$, which we assume to be negative, in front of $\tilde{R}$ which is the trace of the tensor h. The purpose of this section is to show that, up to higher order terms, the same coefficient appears in the evolution of the L^2-norm of the tensor h, which we are trying to kill.

We now introduce the trace free Ricci curvature:

(2.06) $$z_{ij} = R_{ij} - \frac{R}{n} g_{ij}$$

so that $h_{ij} = z_{ij} + \frac{\tilde{R}}{n} g_{ij}$. The standard decomposition of the curvature tensor into irreducible components is then

(2.07) $$R_{ijk}{}^{l} = \frac{R}{n(n-1)} g_{ijk}{}^{l} + Z_{ijk}{}^{l} + W_{ijk}{}^{l},$$

where W is the Weyl conformal curvature tensor, and Z, the traceless Ricci curvature tensor of type $(1, 3)$, is given by

(2.08) $$Z_{ijk}{}^{l} = (z * g)_{ijk}{}^{l},$$

where we define for any tensor s of type $(0, 2)$ the tensor $s * g$ of type $(1, 3)$ by the formula

$$(2.09) \qquad (n-2)(s*g)_{ijk}{}^{l} = s_{jk}g_i^l + g_{jk}s_i^l - s_{ik}g_j^l - g_{ik}s_j^l - \frac{\operatorname{tr} s}{n-1} g_{ijk}{}^{l}$$

with $g_{ijk}{}^{l} = \frac{1}{2}(g*g)_{ijk}{}^{l} = g_{jk}g_i^l - g_{ik}g_j^l$. We then have $(s*g)_{ljk}{}^{l} = s_{jk}$ and the following formula relates the two natural metrics which we use on the tensors of type $(1, 3)$ and $(0, 2)$:

$$(2.10) \qquad (n-2)|s*g|^2 = 2|s|^2 - \frac{1}{n-1}(\operatorname{tr} s)^2.$$

In computing the evolution of the curvature one has to be careful about specifying the type of the tensor, since with respect to a varying metric, not only is the curvature changing but also its projections onto tensors of various types. For example, the evolution equations for z_{ij} are quite different from those for Z_{ijk}^l in the lowest order terms. In order to keep track of the trace free Ricci component, we will denote the corresponding projection by

$$(2.11) \qquad \Pr = \Pr_Z : S_{ijk}{}^{l} \mapsto (t*g)_{ijk}{}^{l},$$

where $S_{ijk}{}^{l}$ is any tensor of type $(1, 3)$, $t_{ij} = s_{ij} - \frac{S}{n} g_{ij}$ with $s_{ij} = S_{kij}{}^{k}$ and $S = g^{ij}s_{ij}$.

The rate of change of this projection map with respect to an infinitesimal variation of the metric given by $\dot g = -2h$ plays an important role in this paper and is computed to be

$$(2.12) \qquad \Pr' = \frac{\partial}{\partial t}\Pr_Z : S_{ijk}{}^{l} \mapsto (t'*g)_{ijk}{}^{l} + T''_{ijk}{}^{l},$$

where $S_{ijk}{}^{l}$ is a fixed tensor of type $(1, 3)$,

$$t'_{ij} = -\frac{2}{n}h^{kl}s_{kl}g_{ij} + \frac{2}{n}Sh_{ij},$$

$$T''_{ijk}{}^{l} = -2h_{jk}t_i^l + 2g_{jk}h^{lm}t_{im} + 2h_{ik}t_j^l + 2g_{ik}h^{lm}t_{jm};$$

both involve only the "trace free Ricci part" t_{ij} of $S_{ijk}{}^{l}$. It follows that

$$\langle t', t\rangle = \frac{2}{n}S\langle h, t\rangle, \qquad |\langle T'', (t*g)\rangle| \le c(n)|h||t|^2,$$

$$\begin{aligned}\langle \Pr'(S), \Pr(S)\rangle &= \langle t'*g, t*g\rangle + \langle T'', t*g\rangle \\ &\le \frac{4}{n-2}\frac{S}{n}\langle h, t\rangle + c(n)|h||t|^2,\end{aligned}$$

where $c(n)$ generically denotes, from now on, any constant depending only on the dimension n.

Applying Pr' to the curvature tensor R^l_{ijk} itself and using the fact that

$$(2.13)\qquad |h|^2 = |z|^2 + \frac{1}{n}\tilde{R}^2 = \frac{n-2}{2}|Z|^2 + \frac{1}{n}\tilde{R}^2,$$

we obtain the following estimate:

$$\begin{aligned}\left\langle \left(\frac{\partial}{\partial t}\mathrm{Pr}_Z\right)(R_{ijk}{}^l),\, Z_{ijk}{}^l\right\rangle &\le \frac{4}{n-2}\frac{R}{n}\langle z,\, z\rangle + c(n)|h||z|^2\\ &= \frac{2}{n}r(0)|Z|^2 + \frac{2}{n}\tilde{R}|Z|^2 + c(n)|h||Z|^2\\ &\le \frac{2}{n}r(0)|Z|^2 + c(n)|h|^3.\end{aligned}$$

The basic evolution equation for the whole Riemannian curvature tensor $\mathrm{Rm} = R_{ijk}{}^l$, regarded as a $T^*M\otimes TM$-valued 2-form as derived in Lemma 4 of [15], is

$$(2.15)\qquad \frac{\partial}{\partial t}\mathrm{Rm} + \Delta^\nabla \mathrm{Rm} + [\mathrm{Rm},\, \mathrm{Rc}] = 0,$$

where $\Delta^\nabla = d^\nabla\delta^\nabla + \delta^\nabla d^\nabla$ is the covariant Laplacian, the Ricci curvature is regarded as a section of $T^*M\otimes TM$ and the second term is defined by $[\mathrm{Rm},\, \mathrm{Rc}]^l_{ijk} = R^l_{ijm}R^m_k - R^l_m R^m_{ijk}$ ($\mathrm{Rc} = \delta_2\mathrm{Rm}$ in the notation of [15]). $[\mathrm{Rm},\, \mathrm{Rc}]$ is therefore a 2-form with values in the *symmetric* endomorphisms of TM, being the Lie bracket of a symmetric and an antisymmetric endomorphism.

The first term is obviously nonnegative in the L^2-sense and the second term, with values in the endomorphisms which are symmetric with respect to the metric g at the given instant in time, is a gauge correction term reflecting the instantaneous change of the orthonormal frame bundle. In particular this last term is always orthogonal (with respect to the metric at any instant) to any 2-form with values in the skew-symmetric endomorphisms, for example any component of the curvature tensor. For our purposes this is a more effective way of describing the changing curvature than the equations given in Hamilton's original paper, which describe the evolution of the curvature tensor of type $(0,4)$. In order to calculate the time derivative of the L^2-norm of the trace free Ricci curvature Z, we need the following:

Basic Lemma. *For any compact Riemannian manifold $(M^n,\, g)$ of dimension $n\ge 3$, we have*

$$(2.16)\qquad \int_M \langle \Delta^\nabla \mathrm{Rm},\, Z\rangle \ge 0.$$

More explicitly, we have the following formulas:

$$(2.17)\quad (n-2)\int\langle\Delta^{\nabla}\mathrm{Rm}, Z\rangle = \int|\delta^{\nabla}\mathrm{Rm}|^2 + \frac{n-4}{2n}\int|dR|^2 \quad \textit{for } n \ge 4,$$

$$(2.18)\qquad \int\langle\Delta^{\nabla}\mathrm{Rm}, Z\rangle = \int|\delta^{\nabla}Z|^2 \quad \textit{for } n = 3.$$

Before we turn to the proof of the Basic Lemma we derive first the fundamental estimates on the time derivative of the L^2-norms of $\tilde{R}$, Z and h as a consequence. First, it follows from multiplying (2.05) by $\tilde{R}$ that

$$(2.19)\qquad \begin{aligned} \frac{1}{2}\left(\frac{\partial}{\partial t}+\Delta\right)\tilde{R}^2 &= -|d\tilde{R}|^2 + 2\tilde{R}|h|^2 + \frac{2}{n}r(0)\tilde{R}^2 \\ &\le \frac{2}{n}r(0)\tilde{R}^2 + (n-2)\tilde{R}|Z|^2 + \frac{2}{n}\tilde{R}^3 \end{aligned}$$

since

$$|h|^2 = |z|^2 + \frac{1}{n}\tilde{R}^2 = \frac{n-2}{2}|Z|^2 + \frac{1}{n}\tilde{R}^2,$$

and hence using $\dot{\mu} = -\tilde{R}\mu$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int\tilde{R}^2 \le \frac{2}{n}r(0)\int\tilde{R}^2 + (n-2)\int\tilde{R}|Z|^2 + \frac{4-n}{2n}\int\tilde{R}^3.$$

On the other hand,

$$\begin{aligned} \int\left\langle\frac{\partial}{\partial t}Z, Z\right\rangle &= \int\left\langle\frac{\partial}{\partial t}(\mathrm{Pr}_Z(\mathrm{Rm})), Z\right\rangle \\ &= \int\left\langle\frac{\partial}{\partial t}(\mathrm{Pr}_Z)(\mathrm{Rm}), Z\right\rangle + \int\left\langle\mathrm{Pr}_Z\left(\frac{\partial}{\partial t}\mathrm{Rm}\right), Z\right\rangle. \end{aligned}$$

The first integral in the last line above is estimated in (2.14), and by (2.15) the last integral is

$$\begin{aligned} \int\left\langle\frac{\partial}{\partial t}\mathrm{Rm}, Z\right\rangle &= -\int\langle\Delta^{\nabla}\mathrm{Rm}, Z\rangle - \int\langle[\mathrm{Rm}, \mathrm{Rc}], Z\rangle \\ &= -\int\langle\Delta^{\nabla}\mathrm{Rm}, Z\rangle + 0 \\ &\le 0 \quad \text{by the Basic Lemma (2.16).} \end{aligned}$$

Therefore

$$\int\left\langle\frac{d}{dt}Z, Z\right\rangle \le \frac{2}{n}r(0)\int|Z|^2 + \frac{2}{n}\int\tilde{R}|Z|^2 + c(n)\int|h||Z|^2,$$

and hence

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int |Z|^2 &\le \int \left\langle \frac{d}{dt}Z\,,\,Z\right\rangle + 2\int |h||Z|^2 - \frac{1}{2}\int \tilde{R}|Z|^2 \\ &\le \frac{2}{n}r(0)\int |Z|^2 + c(n)\int |h||Z|^2 + \left(\frac{2}{n} - \frac{1}{2}\right)\int \tilde{R}|Z|^2 \\ &\le \frac{2}{n}r(0)\int |Z|^2 + c(n)\int |h|^3.\end{aligned}$$

Combining the above two estimates we finally obtain:

Lemma.

$$\frac{1}{2}\frac{d}{dt}\int |h|^2 \le \frac{2}{n}r(0)\int |h|^2 + c(n)\int |h|^3. \tag{2.21}$$

We note that the right-hand side of (2.21) is negative if $r(0) < 0$ and $\max|h|$ is small compared to $-r(0)$. This describes the underlying idea of the whole proof.

Proof of the Basic Lemma. First we remark that the lemma is not obvious, since the covariant Laplacian Δ^{∇} does *not*, in general, respect the decomposition (2.07) of the curvature tensor. The lemma is a consequence of the second Bianchi identity which reflects the invariance of the curvature tensor under the group of diffeomorphisms and is therefore fundamental for any problem related to curvature deformations (compare with Lemma 2 of [15]). The proof we give here is motivated by the interpretation of Hamilton's Ricci flow as a flow of hyperbolic Cartan connections as elaborated in [15], and hence we will be using some of the formulas from that paper. The second Bianchi identity is

$$d^{\nabla}\mathrm{Rm} = 0, \tag{2.22}$$

and therefore, by integration on a compact manifold,

$$\int \langle \Delta^{\nabla}\mathrm{Rm}\,,\,Z\rangle = \int \langle \delta^{\nabla}\mathrm{Rm}\,,\,\delta^{\nabla}Z\rangle.$$

Now by Lemma 2 of [15] or a straightforward calculation,

$$(\delta^{\nabla}\mathrm{Rm})_{ij}^{k} = -d_2^{-1}(d^{\nabla}\mathrm{Rc})_{ij}^{k} = g^{kl}(R_{il,j} - R_{ij,l}), \tag{2.23}$$

where $R_{ij,p} = (\nabla\mathrm{Rc})(e_p\,;\,e_i\,,\,e_j)$ is the covariant derivative of the Ricci tensor.

Similarly by taking the covariant divergence of $Z = z * g$ we obtain

$$(n-2)(\delta^{\nabla}Z)_{ij}^{k} = (\delta^{\nabla}\mathrm{Rm})_{ij}^{k} + \frac{n-4}{2n}(g_i^k R_{\ ,j} - g_{ij}g^{kl}R_{\ ,l}) \tag{2.24}$$

for $n \ge 4$.

Formula (2.17) now follows from the computation:

$$\begin{aligned}(g_{ik}R_{\,,j} - g_{ij}R_{\,,k}) \cdot (R_{ik,j} - R_{ij,k}) &= R_{\,,j}R_{\,,j} - R_{\,,k}R_{jk,j} - R_{\,,j}R_{kj,j} + R_{\,,k}R_{\,,k} \\ &= |dR|^2 - \tfrac{1}{2}|dR|^2 - \tfrac{1}{2}|dR|^2 + |dR|^2 = |dR|^2,\end{aligned}$$

where we have used in an essential manner the contracted second Bianchi identity:

$$R_{ik,k} = \tfrac{1}{2}R_{\,,i} \qquad (R_{\,,i} \text{ means } dR(e_i)). \tag{2.25}$$

For ease of notation, we are using here an orthonormal base $\{e_i\}$ and writing all our indices as subscripts.

For $n = 3$ there is no Weyl curvature in the decomposition (2.07) and we have the simpler formula (2.18) since

$$\Delta^{\nabla}\mathrm{Rm} = \tfrac{1}{6}\Delta R \cdot (g * g) + \Delta^{\nabla} Z \quad \text{and} \quad \langle g * g, Z\rangle = 0.$$

For ready reference, in the next section we record the usual Weitzenböck formulas for the whole Riemannian curvature tensor Rm, which follows from formulas (2.14) and (2.15) of [15] or from Theorem 7.1 of [9]:

$$\Delta^{\nabla}\mathrm{Rm} + [\mathrm{Rm}, \mathrm{Rc}] = \overline{\Delta}\mathrm{Rm} + Q, \tag{2.26}$$

where $\overline{\Delta} = \nabla^*\nabla = -\mathrm{tr}\,\nabla^2$ is the rough Laplacian and

$$\begin{aligned}Q_{ijk}{}^{l} = {} & R_{ij}{}^{pq}R_{pqk}{}^{l} + 2R^{p}{}_{iq}{}^{l}R_{pjk}{}^{q} - 2R^{p}_{jq}{}^{l}R_{pik}{}^{q} \\ & + R_{pjk}{}^{l}R^{p}_{i} + R_{ipk}{}^{l}R^{p}_{j} + R_{ijp}{}^{l}R^{p}_{k} - R_{ijk}{}^{p}R^{l}_{p},\end{aligned}$$

$$\frac{1}{2}\left(\frac{\partial}{\partial t} + \Delta\right)|\mathrm{Rm}|^2 + |\nabla \mathrm{Rm}|^2 + \langle Q, \mathrm{Rm}\rangle = \langle h * \mathrm{Rm}, \mathrm{Rm}\rangle, \tag{2.27}$$

where $(h * \mathrm{Rm})_{ijk}{}^{l} = (g^p_i h^q_j + h^p_i g^q_j)R_{pqk}{}^{l}$.

To derive the analogous equation/inequality satisfied by h we first take the trace of (2.26) to find that Rc satisfies [9, Corollary 7.3]

$$\left(\frac{\partial}{\partial t} + \overline{\Delta}\right)R_{ij} + q_{ij} = 0, \tag{2.28}$$

where

$$\begin{aligned}q_{ij} = Q_{kij}{}^{k} &= 2R_{ipj}{}^{q}R^{p}_{q} + R_{pi}R^{p}_{j} + R_{pj}R^{p}_{i} \\ &= 2R_{ipj}{}^{q}z^{p}_{q} + R_{pi}z^{p}_{j} + R_{pj}z^{p}_{i}.\end{aligned}$$

Since

$$\frac{\partial}{\partial t}h = \frac{\partial}{\partial t}\mathrm{Rc} - \frac{r(0)}{n}g = \frac{\partial}{\partial t}\mathrm{Rc} + \frac{2}{n}r(0)h,$$

and $\overline{\Delta} g = 0$ it follows that

$$\left(\frac{\partial}{\partial t} + \overline{\Delta}\right) h = \frac{2}{n} r(0) h - q, \tag{2.29}$$

where $q = q_{ij}$. (2.29) implies the following inequality:

$$\frac{1}{2}\left(\frac{\partial}{\partial t} + \Delta\right) |h|^2 + |\nabla h|^2 \le c(n)|\mathrm{Rm}||h|^2. \tag{2.30}$$

Finally we remark that the right-hand side of the last estimate can be made more precise since $\langle q, h\rangle = \langle q, z\rangle$ lies between $\lambda_{\min}|z|^2$ and $\lambda_{\max}|z|^2$, where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalues of the *curvature operator* $\hat{R}: \Lambda^2 \to \Lambda^2$. Therefore

$$\frac{1}{2}\left(\frac{\partial}{\partial t} + \Delta\right) |h|^2 + |\nabla h|^2 \le \frac{2}{n} r(0)|h|^2 - \lambda_{\min}|h|^2. \tag{2.31}$$

3. Convergence

We begin by normalizing the initial metric $g(0)$ at $t = 0$. We assume

$$\max |K(0)| = 1, \qquad d(0) \le \Lambda. \tag{3.01}$$

This implies in particular that $r(0) \ge -n(n-1)$.

We set $\lambda = -r(0)/2n > 0$ and define

$$H(t) = \max_{SM} |\rho(t) + 2\lambda|, \tag{3.02}$$

where SM is the set of unit tangent vectors at time t. We have

$$H(t)^2 \le \max |h(t)|^2 \le n \cdot H(t)^2.$$

Our assumption on the initial curvature is therefore

$$H(0) \le -r(0) \cdot \varepsilon = 2n\lambda\varepsilon. \tag{A0}$$

Since $|\mathrm{Rm}|$ and $|h|$ both satisfy parabolic inequalities (2.27) and (2.30) respectively with a quadratic nonlinearity occurring only in the zeroth order terms, the usual maximum principle shows that there exists a universal time $t_0 > 0$, depending only on the dimension n, such that

$$\max |K(t)| \le 10 \cdot \max |K(0)| = 10 \quad \text{for all } t \in [0, t_0], \tag{3.03}$$

$$H(t) \le 5|H(0)| \le 10 n\lambda\varepsilon \quad \text{for all } t \in [0, t_0]. \tag{3.04}$$

This implies $H(t)^2 \le 100 n^2\lambda^2\varepsilon^2 \le 16 n^2\lambda^2\varepsilon \exp(-4\lambda t)$ for $t \in [0, t_0]$, provided $10\varepsilon < c(n) = \exp(-2(n-1)t_0) \le \exp(-4\lambda t_0)$.

To prove that the flow (2.01) can be integrated for all time, provided ε is chosen small enough, we first assume that the following bounds hold for a *maximal* time interval $[0, 2T]$:

$$\text{(A1)} \qquad H(t)^2 \leq 16n^2\lambda^2\varepsilon\exp(-4\lambda t),$$

$$\text{(A2)} \qquad \max|K(t)| \leq 100 \quad \text{for all } t \in [0, 2T].$$

We remark that we have deliberately put in an ε instead of an ε^2 in the estimate A1 for $H(t)^2$ in order to leave some room to prove a contradiction to the maximality of $2T$. It is clear from our initial assumptions that there exists some $2T \geq t_0$ such that the above bounds are valid. Moreover by the main result of [2] or [1], it follows from assumption A2 that

$$\text{(3.05)} \qquad \max|\nabla^k \mathrm{Rm}(t)| \leq C(n, k) \quad \text{for all } t \in [t_0, 2T].$$

We will then show that there exists an $\varepsilon > 0$, depending only on n and Λ, such that these running assumptions A1 and A2, together with the initial assumption A0, would imply that in fact the following stronger curvature bounds actually hold at time $2T$:

$$\text{(B1)} \qquad H(2T)^2 \leq 4n^2\lambda^2\varepsilon\exp(-4\lambda T),$$

$$\text{(B2)} \qquad \max|K(2T)| \leq 20.$$

This would contradict the assumption that $2T$ is maximal, proving that the flow exists for all time and that we have an exponential decay in the C^0-norm for the tensor h, which is the vector field of the flow and, at the same time, measures the deviation from an Einstein metric.

First we observe that assumption A1 implies a uniform bound on all the metrics $g(t)$ for $t \in [0, 2T]$. This is because the change in the metric satisfies

$$\text{(3.06)} \qquad \max_{|v|=1}\left|\frac{d}{dt}\log g_t(v, v)\right| \leq H(t),$$

and hence

$$\text{(3.07)} \qquad \exp(-2n\sqrt{\varepsilon}) \leq \max_{|v|=1}\frac{g_t(v, v)}{g_0(v, v)} \leq \exp(2n\sqrt{\varepsilon}),$$

since, by A1,

$$\int_0^{2T} H(t) \leq 4n\lambda\sqrt{\varepsilon}\int_0^{2T}\exp(-2\lambda t) \leq 2n\sqrt{\varepsilon}.$$

(3.07) implies the following volume estimate:

$$\text{(3.08)} \qquad \exp(-n^2\sqrt{\varepsilon}) \leq \frac{V(t)}{V(0)} \leq \exp(n^2\sqrt{\varepsilon}).$$

Next we derive an L^2-estimate for the tensor h. From inequality (2.21) we obtain

$$\frac{d}{dt}\int |h|^2 \leq -8\lambda \int |h|^2 + c(n)H\int |h|^2. \tag{3.09}$$

Using the notation $\|\ \|_2$ for the L^2-norm, we have from our assumption A1 the following inequality:

$$\frac{d}{dt}\|h(t)\|_2^2 \leq -8\lambda\|h(t)\|_2^2 + c(n)\lambda\sqrt{\varepsilon}\|h(t)\|_2^2 \leq -4\lambda\|h(t)\|_2^2 \tag{3.10}$$

if we choose $\varepsilon < \varepsilon(n)$. This implies the L^2-estimate:

$$\|h(t)\|_2^2 \leq \|h(0)\|_2^2 \exp(-4\lambda t) \quad \text{for } t \in [0, 2T]. \tag{3.11}$$

In order to proceed to a C^0-estimate we use the parabolic inequality (2.30):

$$\frac{1}{2}\left(\frac{\partial}{\partial t} + \Delta\right)|h|^2 + |\nabla h|^2 \leq c(n)|\mathrm{Rm}||h|^2.$$

Now applying the Cauchy-Schwarz inequality, dividing through by $|h|$, and using the uniform bound A2 on the whole curvature tensor Rm we obtain

$$\left(\frac{\partial}{\partial t} + \Delta\right)|h| \leq c(n)|h| \quad \text{for } t \in [0, 2T], \tag{3.12}$$

where of course the inequality is to be understood in the weak distributional sense at the points where $h = 0$. It follows therefore that for any $q \geq 1$

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \Delta\right)|h|^q &= q|h|^{q-1}\left(\frac{\partial}{\partial t} + \Delta\right)|h| - q(q-1)|h|^{q-2}\cdot|d(|h|)|^2 \\ &\leq c(n)\cdot q|h|^q.\end{aligned}$$

By setting $v_0 = |h|$ and $v_{k+1} = v_k^p$ for $k = 0, 1, \cdots$ with $p = (n+2)/n$ we obtain

$$\left(\frac{\partial}{\partial t} + \Delta\right)v_k \leq c(n)p^k\cdot v_k, \qquad k = 0, 1, \cdots, \infty. \tag{3.13}$$

If a nonnegative function u satisfies $\frac{\partial}{\partial t}u + \Delta u \leq A\cdot u$, and χ is a function of t alone, then we have

$$\begin{aligned}&\frac{d}{dt}(\chi^2\|u\|_2^2) + 2\chi^2\|du\|_2^2 - 2\chi\dot{\chi}\|u\|_2^2 \\ &\quad = 2\chi^2\int(u\cdot\dot{u} + u\cdot\Delta u) + \chi^2\int u^2 \operatorname{tr} h \leq \chi^2(2A + nH)\|u\|_2^2.\end{aligned} \tag{3.14}$$

For any $0 < t_1 < t_1 + \tau \le t_2 \le 2T$, we choose a cut-off function $\chi(t)$ satisfying $\chi \equiv 0$ on $[0, t_1]$, $\chi \equiv 1$ on $[t_1 + \tau, \infty]$ and $0 < \dot{\chi} < 2\tau^{-1}$. By integrating the above inequality over the interval $[0, t_2]$, neglecting the energy term $\|du\|_2^2$, and choosing ε so that $nH(t) \le 4n^2\lambda\sqrt{\varepsilon} \le 2n^2(n-1)\sqrt{\varepsilon} \le 2A$, we obtain

$$\|u(t_2)\|_2^2 \le 4(A + \tau^{-1}) \int_{t_1}^{t_2} \|u(t)\|_2^2. \tag{3.15}$$

By neglecting the first term we also get the energy estimate

$$\int_{t_1+\tau}^{t_2} \|du(t)\|_2^2 \le 4(A + \tau^{-1}) \int_{t_1}^{t_2} \|u(t)\|_2^2. \tag{3.16}$$

Following Moser [16] we would like to combine these two estimates via the Sobolev inequality:

$$\|u\|_{2m}^2 \le C_{\mathrm{Sob}} \|du\|_2^2 + c(n)\|u\|_2^2, \tag{3.17}$$

where $m = n/(n-2)$, to obtain an estimate of the form

$$\int_{t_1+\tau}^{t_2} \|u^p\|_2^2 \le c(n) C_{\mathrm{Sob}} \left((A + \tau^{-1}) \int_{t_1}^{t_2} \|u(t)\|_2^2 \right)^p. \tag{3.18}$$

However, since our metric is changing in time, we first have to establish a uniform bound for the *Sobolev constant* C_{Sob}. It is by now well known that the best constants in Sobolev inequalities are determined by the *isoperimetric constant* defined by

$$C_{\mathrm{Iso}} = \inf\{(\mathrm{vol}\,\partial D)^n / (\mathrm{vol}\, D)^{n-1}\},$$

where the infimum is taken over all (not necessarily connected) open submanifolds $D^n \subset M^n$ with smooth boundary ∂D^{n-1} and $2 \cdot \mathrm{vol}(D) \le \mathrm{vol}(M)$. The precise relation with the optimal C_{Sob} appearing in (3.19) is then $C_{\mathrm{Sob}} = c(n) C_{\mathrm{Iso}}^{-2/n}$ (see [13], [6]).

By its very definition C_{Iso} is a C^0-invariant of the metric, and by our uniform estimate (3.05) for the relative C^0-norms of the metrics we have $C_{\mathrm{Iso}}(0) \le c(n) \cdot C_{\mathrm{Iso}}(t)$ for $t \in [0, 2T]$. Now by results of S. Gallot [6, Theorem 1.1] or [3], which are based on an isoperimetric inequality due to M. Gromov [8], we know that $C_{\mathrm{Iso}} V^{-1}$ can be bounded from below by a constant depending only on an upper bound for the diameter and a lower bound for the Ricci curvature (scaled with the square of the diameter). Since we also have a uniform bound (3.08) for the volume change we get

$$C_{\mathrm{Sob}}(t) \le C(n, \Lambda) V(0)^{-2/n} \quad \text{uniformly for all } t \in [0, 2T], \tag{3.19}$$

where $V(0)$ is the initial volume and $C(n, \Lambda)$ depends only on n and Λ.

We can now follow Moser and apply the basic iteration step (3.14)–(3.18) to the sequence of power functions v_k satisfying (3.13), with an appropriate choice of cut-off functions χ_k. We do not have to worry about cutting off with respect to the space variables as in [16] since we are on a compact manifold without boundary. Taking the limit as $k \to \infty$ then gives the following C^0-estimate (for details we refer to the original paper [16, Theorem 3, pp. 113–117]):

$$\begin{aligned}\max_M |h(2T)|^2 &\le c(n) \cdot T^{-p} \left(\max_{[T,2T]} C_{\mathrm{Sob}}(t) \right)^{n/2} \int_T^{2T} \|h(t)\|_2^2 \\ &\le C(n, \Lambda) V(0)^{-1} T^{-p} \int_T^{2T} \|h(t)\|_2^2.\end{aligned} \tag{3.20}$$

Using now the fact that T is estimated from below by a universal bound, $2T \ge t_0 > 0$, with t_0 depending only on n, and substituting the basic L^2-estimate (3.11) in (3.20) we get

$$\begin{aligned}\max_M |h(2T)|^2 &\le C(n, \Lambda) V(0)^{-1} T^{-p} \int_T^{2T} \|h(0)\|_2^2 \exp(-4\lambda t) \\ &\le C(n, \Lambda) \cdot H(0)^2 ⨍_T^{2T} \exp(-4\lambda t) \\ &\le C(n, \Lambda) \lambda^2 \varepsilon^2 \exp(-4\lambda T) \quad \text{by A0},\end{aligned} \tag{3.21}$$

where again we write $⨍$ for the average value, and $C(n, \Lambda)$ for any generic constant depending on n and Λ. Because we have generously chosen ε instead of ε^2 in A1, $\exists \varepsilon > 0$ depending on n and Λ, such that

$$H(2T)^2 \le C(n, \Lambda) \lambda^2 \varepsilon^2 \exp(-4\lambda T) < 4n^2 \lambda^2 \varepsilon \exp(-4\lambda T).$$

This establishes the bound B1 in contradiction to the maximality of the interval $[0, 2T]$ for A1 to hold.

To obtain higher order estimates for h, we regard equation (2.29) as a *linear* parabolic system with coefficients depending on the changing metric $g(t)$ and its derivatives. Under our running assumptions A1 and A2, we have a uniform bound on the metric (3.07) and, by our smoothing result [2], also a uniform bound (3.05) on the derivatives of the curvature in *a smaller time interval* $[t_0, 2T]$. This implies that with respect to a good local coordinate system (normal coordinates would do) the coefficients appearing in (2.29) are bounded in the C^2-norm by some universal constants $c(n)$ after some universal time t_0 has elapsed. Thus we can apply standard interior regularity estimates for linear (or quasilinear) parabolic systems to obtain bounds on the higher derivatives of the tensor h, in terms of its C^0-norm in a normal coordinate neighborhood around any given point on M. We refer to [12, Chapter VII, Theorems 3.1, 4.1, 5.1 and 5.2] for such local estimates. Since every point of M is an interior point and, in normal coordinates, covariant derivatives coincide with the partial derivatives at the origin of the coordinate system, we obtain a uniform global C^2-estimate:

(3.22)

$$\|h(t)\|_{C^2} \le c(n)\|h(t)\|_{C^0} \le c(n)\lambda\sqrt{\varepsilon}\exp(-2\lambda t) \quad \text{for all } t \in [t_0, 2T].$$

Using now a formula (see, e.g., [4, Theorem 1.174]) for the rate of change of the curvature tensor in terms of the derivatives of $\frac{d}{dt}g = -2h$, we have an estimate of the form

$$\left|\frac{d}{dt}\mathrm{Rm}\right| \le c(n)\cdot(|\nabla^2 h| + |\mathrm{Rm}||h|). \tag{3.23}$$

Hence by (3.22),

$$\left|\frac{d}{dt}\mathrm{Rm}\right| \le c(n)\|h\|_{C^2} \le c(n)\lambda\sqrt{\varepsilon}\exp(-2\lambda t) \quad \text{for } t \in [t_0, 2T].$$

Therefore

$$\begin{aligned} \max|K(2T)| &\le \max|K(t_0)| + c(n)\sqrt{\varepsilon}\int_{t_0}^{2T} \lambda\exp(-2\lambda t) \\ &\le 10 + c(n)\sqrt{\varepsilon} \quad \text{by (3.03)} \\ &\le 20 \quad \text{for some } \varepsilon(n) > 0. \end{aligned}$$

This establishes the bound B2. We have thus proved that there exists an $\varepsilon(n, \Lambda) > 0$ such that if the initial Ricci curvature satisfies A0 then the running assumptions A1 and A2 together with all the ensuring estimates which we derived above are in fact valid for $T = \infty$. Since all the relevant bounds on the tensor h and its derivatives are exponentially decaying, we can now refer to Hamilton's original paper [9, Corollary 17.10] to conclude that if the initial metric $g(0)$ satisfies A0, with $\varepsilon(n, \Lambda) > 0$ determined above, then the flow of metrics defined by the vector field $\frac{d}{dt}g = 2h$ can be integrated for all time converging smoothly in the limit as $t \to \infty$ to an Einstein metric $\overline{g} = g(\infty)$ with $\rho(\overline{g}) = r(0)/n = -2\lambda < 0$, proving the theorem.

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